($\omega = 200\pi$ sec^{-1}, $a_2 = 0$, $a_1 = 1$) had no significant effect on the form or orientation of the rotor's steady-state trajectories. Compared to the case when perturbations are absent (see Fig. 2), the presence of finite vertical perturbations leads to an increase in the amplitude of the steady-state trajectories by a factor of 8-9, a shift in the center of the trajectory from the first to the third quadrant in the (x, y) plane, and transformation of the originally circular path into an elliptical path. Most of our computer calculations were performed on 16 × 40 and 24 × 60 grids.

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MOTION OF A BED LOAD

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This article deals with the problem of mathematically modeling a thin moving layer of a two-phase mixture bounded below by a stationary granular medium and above by a liquid flow. The position of the top boundary of the moving layer is specified beforehand along with the normal and shear stresses on it. These characteristics can be obtained from the solution of hydrodynamic equations.

The moving mixture is assumed to be uniform and its acceleration small (the latter is ignored). As is customary in calculations performed for shallow water [1], the contribution of shear stresses on areas normal to the surface is considered to be negligible, while pressure is distributed in accordance with a hydrostatic law.

Formulation of the Problem. In accordance with the above assumptions, the equations of motion are written in the form

$$\begin{gathered}\partial p/\partial s + \rho g \partial\xi/\partial s + \partial\tau_s/\partial m = 0,\\ \partial p/\partial l + \rho g \partial\xi/\partial l + \partial\tau_l/\partial m = 0,\ \partial p/\partial m = \rho g \cos\gamma,\end{gathered} \tag{1}$$

where $\xi = \xi(x, y)$ is the equation of the surface of the mixture; x and y are horizontal cartesian coordinates; s and ℓ are orthogonal curvilinear coordinates on the surface of the mixture; m is the axis directed along a normal to the surface of the mixture (m = 0 on the surface $\xi = \xi(x, y)$); p is the pressure in the mixture; τ_s and τ_ℓ are projections of the shear stress $\boldsymbol{\tau}$ on areas parallel to the surface of the mixture; ρ is the density of the mixture ($\rho = f\rho_r + (1 - f)\rho_b$); ρ_r and ρ_b is the density of the particles and water; f is concentration, the value of which is determined below; γ is the acute angle between the normal to the surface of the mixture and a vertical line.

The rheological relation for the shear stress includes Coulomb's law for a bulk medium and Prandtl's law for a fluid:

$$\boldsymbol{\tau} = -\left(\frac{\partial \mathbf{u}}{\partial m}\Big/\left|\frac{\partial \mathbf{u}}{\partial m}\right|\right)(p_s \operatorname{tg}\varphi + \tau_b). \tag{2}$$

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Here, p_s is the additional pressure in the mixture due to the particles; $p_s = \rho_s gm \cos \gamma$; $\rho_s = \rho - \rho_b = f(\rho_r - \rho_b)$; ϕ is the angle of internal friction (taken equal to 28°, as in the case of a still mixture); $\tau_b = \rho_b L^2 |\partial u/\partial m|^2$; L is the mixing length ($L = \kappa(a - m)$); a is the thickness of the layer of moving mixture; κ is the Kármán constant. According to the tests performed by Nikuradze for pure water, this constant has a value of 0.4. For the mixture in [2], the value of the constant is lower and depends on concentration. For example, $\kappa = 0.2$ at $f = 0.2$-0.3.

In a moving layer of the mixture ($m \le a$), it follows that $|\boldsymbol{\tau}| = \rho_s \tan \phi + \tau_b$, while in a stationary (bottom) layer ($m \ge a$) $|\boldsymbol{\tau}| \le p_s \tan \phi$. Thus, by virtue of the continuity of $|\boldsymbol{\tau}|$, it is necessary to satisfy the following condition on the bottom boundary of the layer

$$\tau_b = 0 \text{ at } m = a. \tag{3}$$

The velocity continuity condition must also be satisfied. From this,

$$\mathbf{u} = 0 \text{ at } m = a. \tag{4}$$

The shear and normal stresses on the top boundary ($m = 0$) are assigned

$$\boldsymbol{\tau} = \boldsymbol{\tau}_\xi,\ p = p_\xi \text{ at } m = 0. \tag{5}$$

Thus, we have formulated boundary-value problem (1)-(5) for determining unknown functions of the thickness of the layer a and the velocity vector **u**.

A special case of Eq. (2), for uniform unidimensional motion of the mixture at $\xi = 0$, was used in [3] to determine the rate of particle flow. Boundary-value problem (1)-(5) was used in [4, 5] to describe the motion of a mixture with a viscous friction law for τ_b, and it was used in [6] with additional assumptions regarding the value of L. A survey of studies of the mechanics of granulated media can be found in [7].

Solution of a Unidimensional Problem. We will examine a unidimensional problem concerning motion along the s axis. We find from the last equation of system (1) that $p = p_\xi + \rho gm \cos \gamma$. With a hydrostatic pressure distribution in the case when the slope of the free surface is much less than the slope of the bottom,

$$\partial p/\partial s + \rho g \partial \xi/\partial s = A\Gamma\ (A = \rho_s g \operatorname{tg} \varphi \cos \gamma,\ \Gamma = \partial \xi/\partial s/(\operatorname{tg} \varphi \cos \gamma)).$$

Integrating the first equation of system (1) over m, we have

$$\tau = \tau_\xi - m\Gamma A. \tag{6}$$

With allowance for boundary condition (3), we obtain the following from Eqs. (2) and (6) at $m = a$

$$a = \tau_\xi/[A(1 + \Gamma)]. \tag{7}$$

It is obvious that the condition of smallness of the accelerations can be used at $|\Gamma| < 1$.

The particles are acted upon by drag from the direction of the liquid, the force of interaction between one particle and another, and the weight of the particles and liquid. We ignore acceleration. Using the condition of equilibrium of the particle is an elementary volume $\Delta m \Delta s$, we obtain

$$Nc_x \rho_b v^2 d^2/2 = \Delta m \Delta s \partial \tau_k/\partial m + A\Gamma \Delta m \Delta s,$$

where v is the difference in the velocities of the mixture and the moving particles; N is the number of particles in the volume $\Delta m \Delta s$; $\tau_k = Am$ is the friction of the particles; $c_x \approx 1$ is the particle drag coefficient; d is particle diameter. Since $Nd^2 = f\Delta m \Delta s/d$, after performing the appropriate transformations

$$v = \sqrt{\frac{2d}{c_x f \rho_b}} \sqrt{A(1 + \Gamma)} = \sqrt{\frac{2d}{c_x f \rho_b}} \sqrt{\frac{\tau_\xi}{a}}.$$

We have the following from Eqs. (2), (6), and (7)

$$\left|\frac{\partial u}{\partial m}\right|^2 = \frac{\tau_\xi - mA(1 + \Gamma)}{\rho_b [\varkappa (a - m)]^2} = \frac{\tau_\xi}{\rho_b \varkappa^2 a(a - m)}.$$

Thus, $\frac{\partial u}{\partial m} = -\frac{\sqrt{\tau_\xi}}{\sqrt{\rho_b}\,\varkappa\sqrt{a}\sqrt{a-m}}$.

We calculate the mass rate of particle flow through integration by parts, taking into account boundary condition (4):

$$G = f\rho_r\int_0^a u\,dm - f\rho_r\int_0^a v\,dm = f\rho_r um\Big|_0^a -$$

$$- f\rho_r\int_0^a m\frac{\partial u}{\partial m}\,dm - f\rho_r va = f\rho_r\sqrt{\frac{\tau_\xi}{\rho_b}}\frac{4}{3}\frac{a}{\varkappa}\left(1-\frac{3\varkappa}{4}\sqrt{\frac{2}{c_x f}}\sqrt{\frac{d}{a}}\right).$$

We use Eq. (7) to find

$$G = \frac{G_0\left(1-\frac{3\varkappa}{4}\sqrt{\frac{2}{c_x f}}\sqrt{\frac{d}{a}}\right)}{1+\Gamma}, \tag{8}$$

where

$$G_0 = \frac{4}{3}\frac{\rho_r\tau_\xi^{1,5}}{\varkappa\sqrt{\rho_b}(\rho_r-\rho_b)g\,\mathrm{tg}\,\varphi\cos\gamma}.$$

The problem of the critical shear stresses on particles τ_* has been closely examined in hydraulics. In a mixture, these stresses should be m = a = d. Using (7) with $\Gamma = 0$ and $\cos\gamma = 1$ we obtain the formula $\tau_* = Ad = f(\rho_r - \rho_b)g\tan\phi d$. In most cases, this formula coincides with f = 0.07. If we make use of the well-known relation between the shear stress on the bottom and the square of the velocity of the top liquid flow averaged over the depth, we can use the relation for τ_* to obtain a formula for the eroding velocity. This result also agrees well with existing experimental data and empirical formulas and agrees completely with the formula obtained by V. A. Goncharov.

At $\kappa = 0.25$, the values of G will be close to those found from the formula in [8, 9]. As was noted in [2, 10], they describe the experimental data well. It should be noted that at $\kappa = 0.25$, f = 0.07 and $c_x = 1(3\kappa/4)\sqrt{2/(c_x f)} = 1$, we can also simplify (8): $G = G_0(1 - \sqrt{d/a})/(1+\Gamma)$. Here, G = 0 at d = a or $\tau_\xi = \tau_*$. The factor $(1 - \sqrt{\tau_*/\tau_\xi})$ is present in most empirical formulas for G, which is additional evidence of the reliability of the results. Thus, the values of the empirical parameters are determined: $\kappa = 0.25$; f = 0.07.

Solution of Two-Dimensional Problem. We introduce the vector $\mathbf{\Gamma}$ with the projections $\Gamma_s = (\partial\xi/\partial s)/(\tan\phi\cos\gamma)$, $\Gamma_l = (\partial\xi/\partial l)/(\tan\phi\cos\gamma)$. Then with a hydrostatic pressure law, by analogy with the unidimensional case

$$\partial p/\partial s + \rho g\partial\xi/\partial s = A\Gamma_s,\ \partial p/\partial l + \rho g\partial\xi/\partial l = A\Gamma_l.$$

Integrating the first two equations of system (1) over m, we obtain

$$\mathbf{F} = \boldsymbol{\tau}\ (\mathbf{F} = \boldsymbol{\tau}_\xi - mA\mathbf{\Gamma}). \tag{9}$$

We will assume that the s axis is directed along τ_ξ.

Having equated the squares of the moduli of the left and right sides of Eq. (9) with m = a, with allowance for boundary condition (3) and rheological relation (2), we obtain a quadratic equation relative to the unknown thickness of the moving layer a. There is only one positive root of this equation

$$a = \frac{\tau_\xi}{A}\frac{\sqrt{1-\Gamma_l^2}-\Gamma_s}{1-\Gamma^2}\quad (\Gamma^2 = \Gamma_s^2 + \Gamma_l^2). \tag{10}$$

For small Γ

$$a = (\tau_\xi/A)(1 - \Gamma_s). \tag{11}$$

With a particle diameter d = a, we can use Eq. (10) to obtain the critical shear stresses of the particles

$$\tau_* = [A(1-\Gamma^2)\,d]\Big/\left(\sqrt{1-\Gamma_l^2} - \Gamma_s\right).$$

This expression coincides with the well-known expression used in practice to project the erosion of channels with $\Gamma_s = 0$. It follows from (2) and (9) that

$$\left|\frac{\partial \mathbf{u}}{\partial m}\right| = \left[\frac{|\mathbf{F}| - mA}{\rho_b\,[\varkappa(a-m)]^2}\right]^{0,5}, \quad \frac{\partial \mathbf{u}}{\partial m} = -\frac{\mathbf{F}}{|\mathbf{F}|}\left|\frac{\partial \mathbf{u}}{\partial m}\right|.$$

By analogy with the unidimensional case, we allow for the relative motion of the particles and use integration by parts to determine the unit rate of flow of solid particles:

$$\mathbf{G} = -f\rho_r\left(1 - \sqrt{\frac{d}{a}}\right)\int_0^a m\frac{\partial \mathbf{u}}{\partial m}\,dm = \frac{f\rho_r}{\sqrt{\rho_b}\,\varkappa}\left(1 - \sqrt{\frac{d}{a}}\right)\int_0^a \frac{m}{a-m}\frac{\mathbf{F}}{|\mathbf{F}|}(|\mathbf{F}| - mA)^{0,5}\,dm. \tag{12}$$

At small Γ, the following series expansion is valid

$$\mathbf{G} = G_0\left[\frac{\tau_\xi}{|\tau_\xi|}(1 - 0{,}2\Gamma_s) - 0{,}8\Gamma\right]\left(1 - \sqrt{\frac{d}{a}}\right). \tag{13}$$

Let us examine flow in a channel with uniform motion of water at a constant flow rate Q. SInce erosion takes place slowly, it can be assumed that τ_ξ is determined at each moment of time by the relation for unidimensional motion $\tau_\xi = \rho_b ghI$ (where h is depth and I is the inclination of the free surface). The pump balance in this equation is written in the form

$$\frac{\partial \xi}{\partial t} + \frac{1}{\rho_r(1-\varepsilon)}\frac{\partial G_y}{\partial y} = 0, \tag{14}$$

where ε = 0.4 is porosity; G_y is the projection of **G** on the horizontal axis y, which is directed perpendicular to water flow velocity.

Figure 1 shows the change in the position of bottom markers in a cross section of a chute (with Q = 0.112 m³/sec and t = 0.3, 2.5, and 5.54, respectively, in a-c) (the tests were conducted in the hydraulics laboratory of the TsNIIS (All-Union Scientific Research Institute of Transportation Engineering) by A. N. Militeev and N. L. Moizhes. The points represent experimental values, the solid line shows the surface of the bottom obtained from calculations with Eq. (14), the bottom dashed line shows the position of the bottom at the initial moment of time t = 0, and the top horizontal line (dashed) shows the position of

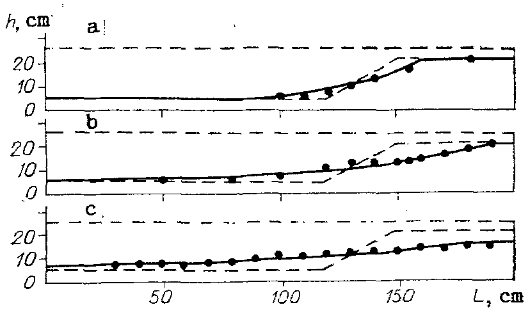

Fig. 1

the free surface of the water. The diameter of the soil particles was 0.2 mm in all of the tests. Calculations of G_y were performed with Eqs. (10) and (12), as well as with (11) and (13). The results were close in both cases and coincided in graph form.

To check the measured value of I, we determined the roughness coefficient in the Chezy–Manning formula. It turned out to be 0.02 in each case, corresponding to the tabulated value in [11].

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